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METHOD OF TRANSLATIONS FOR A MODE I ELLIPTIC CRACK IN AN INFINITE BODY. PART I: POLYNOMIAL LOADING

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Abstract—A method has been proposed for determining the displacement of the elliptic crack faces in an infinite body and consequently stress intensity factors under the action of polynomial loading. The method is based: on the Rice integral formula which relates the stress and displacement fields for two different states of a body; on Dyson's theorem which defines the form of the displacement field for the prescribed law of the action of the polynomial loading; on the theory of the elliptic crack translations in a nonuniform stress field developed in the present study; and finally on the known solution for a uniform loading.

The method proposed does not require the solution of boundary problem and actually represents itself the recurrent procedure for step by step determination of the displacement field for higher and higher degrees of polynomial loading. In its structure, objectives and complexity the method corresponds to the weight function methods known in the literature whose main feature is the use of the known particular solutions for the given body in order to obtain new solutions. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

In spite of an abundance of computation software for calculating stress intensity factors, K_I, for 3-D bodies with a mode I crack [see, for example, FEM-based ones developed by Raju and Newman (1982), Shiratori (1986)], the development of theoretical methods and obtaining exact analytical solutions remains a topical problem. They are necessary:

(1) to verify the exactness of the computational software or approximate fundamental solutions for K_I (see, for example, Oore and Burns, 1980; Orynyak *et al.*, 1994);
(2) to develop the theory itself whose methods can be applied to solve other problems;
(3) to develop combined methods which allow K_I determination in a body with a crack on the basis of more simple FEM-based calculations for the same body without crack (see, for example, Nishioka and Atluri, 1982; Nishioka and Atluri, 1983).

From the standpoint of mathematics the problem of K_I determination is reduced to the solution of the problem of the potential theory and the fundamental integrodifferential equation has the following form (see, for example, Walpole, 1970):

$$p(x,y) = L\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\iint\limits_{(S)}\frac{U(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta}{\sqrt{(x-\xi)^2-(y-\eta)^2}} \tag{A}$$

where (x,y) and (ξ,η) are Cartesian coordinates in the crack plane related to the centre of

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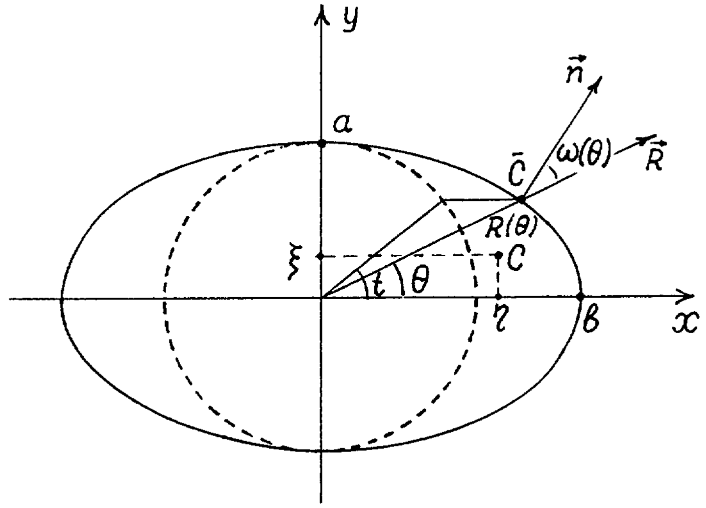

Fig. 1. The nomenclature of elliptical crack.

the elliptic crack (Fig. 1); S is the crack area; L is some constant related to the material elastic constants; $p(x, y)$ is the load acting on the crack faces; $U(\xi, \eta)$ are the displacements of the crack surface points. Displacements $U(\xi, \eta)$ are to defined for the given loading of the crack faces.

For infinite bodies fundamental solutions [i.e. for any $p(x, y)$ law] exist only for a semi-infinite crack with a straight front (see Uflyand, 1968) and for a penny-shaped crack (Galin, 1953). So far there is no such fundamental solution for an elliptic crack, though in the case of the polynomial law of load distribution on the crack faces there exist two radical approaches to the solution of the problem.

The first approach originates from the works of Green and Sneddon (1950) and Kassir and Sih (1966), who obtained analytical solution for an elliptic crack subjected to uniform normal and shear loading, respectively. It is based on the solution of a boundary problem for a half-space in the formulation of Trefftz (1928) and it involves the introduction of the Lamé ellipsoidal function. Later on Segedin (1967) proposed a general form of the harmonic potential function for the case of polynomial loading of the crack faces. Those results were used in subsequent researches of Shah and Kobayashi (1971) who obtained a solution for normal polynomial loading up to the third degree, Vijayakumar and Atluri (1981) who presented a general procedure of the solution for an elliptic crack loaded by arbitrary normal and shear stresses.

The second approach is based on the use of the old theorem of Dyson (1891), the results of which were repeated later independently in the works of Galin (1947) and Kassir and Sih (1966). This theorem was used in the works of Borodachev, A. N. (1981) and Borodachev, N. M. *et al.* (1992) who repeated some results of Shah and Kobayashi (1971) and obtained a general structure of solution (but not the solution itself) for the crack faces loaded up to the fourth degree. For cracks Dyson's theorem is reduced to the following: if polynomial loading of the crack faces $p(x, y)$ is prescribed

$$p(x,y) = P_{ij}\left(\frac{y}{a}\right)^i \left(\frac{x}{b}\right)^j \tag{B}$$

the displacement $U(\xi, \eta)$ of an arbitrary point C (Fig. 1) on the upper crack surface has the following form:

$$u(\xi,\eta) = \frac{P_{ij}\cdot a}{H}\Omega \sum_{m+n=0}^{m+n=i+j} q_{m,n}^{i,j}\left(\frac{\xi}{a}\right)^m \left(\frac{\eta}{b}\right)^n, \tag{C.1}$$

where for convenience it is designated

$$\sqrt{1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}}=\Omega \tag{C.2}$$

Here H is the Young modulus; a, b are the ellipse semi-axes; $q^{i,j}_{m,n}$ are some coefficients depending on the ratio of the ellipse semi-axes $\lambda = a/b \leqslant 1$. Coefficients $q^{i,j}_{m,n}(\lambda)$ are determined by substituting expressions (B) and (C.1) into the integrodifferential eqn (A) of the elasticity theory. Then in an arbitrary point of the crack contour $\bar{C}$ the stress intensity factor $K_{\rm I}$ is equal to (see, for example, Sih and Liebowitz, 1968):

$$K_{\rm I}(t) = p_{ij}\sqrt{\pi a}\Pi^{1/4}(t)\sum_{m+n=0}^{m+n=i+j} q^{i,j}_{m,n}(\lambda)\sin^m t\cos^n t \tag{D.1}$$

or

$$K_{\rm I}(t) = p_{ij}\sqrt{\pi a}\Pi^{1/4}(t)\bar{K} \tag{D.2}$$

where $\bar{K}$ is the dimensionless stress intensity factor, t is the parametric angle of the contour point $\bar{C}$, with $tg\theta = \lambda tgt$ (Fig. 1) and

$$\Pi(\theta) = (\sin^2 t+\lambda^2\cos^2 t) \tag{D.3}$$

This approach was first used by Panasyuk (1968) who obtained the solution of integrodifferential eqn (A) for a uniform stress. In spite of the fact that the use of Dyson's theorem eliminates the necessity of using a cumbersome apparatus of Lamé ellipsoidal functions and Jacobian elliptic functions, yet the method is rather complicated for obtaining new solutions for $K_{\rm I}$ in practice.

The goal of the present paper is to develop the third, alternative approach to the solution of the problem for an elliptic crack with the polynomial loading of its faces. The method proposed does not require the solution of boundary problem and actually represents itself the recurrent procedure for step by step determination of the displacement field for higher and higher degrees of polynomial loading. The methods is applicable only to infinite geometry. For finite medium other methods as Martin's (1986) polynomial expansion method and Roy and Chatterjee's (1994) integral equation method have wider use.

2. THE IDEAS BORROWED FROM THE LITERATURE

In its structure, objectives and complexity the method corresponds to the engineering weight function methods known in the literature whose main feature is the use of the known particular solutions for a given body in order to obtain new solutions (see, for example, Petroski and Achenbach, 1978; Fett, 1988). The method involves the following known results:

(1) Dyson's theorem the essence of which is given above. As it follows from the theorem, if the displacements of the surface points are presented in the form:

$$U(\xi,\eta) = \frac{q_{mn}}{H}a\Omega\left(\frac{\xi}{a}\right)^m\left(\frac{\eta}{b}\right)^n \tag{1}$$

the loads corresponding to them are a polynomial series

$$p(x,y) = q_{mn} \sum_{i+j=0}^{i+j=m+n} p_{i,j}^{m,n} \left(\frac{y}{a}\right)^i \left(\frac{x}{b}\right)^j. \quad (2)$$

For this reason two equivalent formulations of the objective of the present work can be given:

(a) to determine coefficients $q_{m,n}^{i,j}$ of the displacement expansion in formula (C.1) for the given law of the crack faces loading (B);

(b) to determine coefficients $p_{m,n}^{i,j}$ of the stress expansion in formula (2) for the given law of the crack faces displacement (1).

(2) Rice's formula of energy balance (Rice, 1972), which relates integrally the stress intensity factors (for two different states of a body) with the applied loading for the first state and with the displacement field for the second one:

$$\frac{1}{2}\int_{\Gamma} \frac{K_{\mathrm{I}}^1 K_{\mathrm{I}}^2}{H} \cdot \partial l \,\mathrm{d}\Gamma = \iint_{(S)} p^1 \delta U^2 \,\mathrm{d}S \quad (3)$$

where the upper indices refer to the first and second states, respectively; δU^2 is the virtual variation of the displacement field induced by the virtual translation of the crack front δl; S is the crack area; Γ is the length of the crack contour. It is important to note that the derivation of formula (3) involved the assumption that the stress field did not change during the translation of the crack.

(3) The solution of Green and Sneddon (1950) for uniform loading, i.e. if

$$p(x,y) = p_{00}, \quad (4a)$$

then

$$U(\xi,\eta) = \frac{p_{00}a}{HE(k)} \cdot \Omega \quad (4b)$$

$$K_{\mathrm{I}}(\theta) = \frac{\Pi^{1/4}(\theta)\sqrt{\pi a}}{E(k)} \quad (4c)$$

where $E(k)$ is complete elliptic integral of the second kind, $k^2 = 1-\lambda^2$.

(4) The notion of the crack front points translation which retains the ellipticity of the crack shape (see Cruse and Besuner, 1975). These types of translations are presented in the most ordered fashion in the works of Vainshtok (1988) and Vainshtok and Varfolomeyev (1990). Retaining mainly their designations, we shall write with some adjustments:

$$\partial l = \cos\omega\delta R = \cos\omega \frac{R(\theta)}{a}\mu_s \cdot \delta T_s \quad s = 1,\ldots,5 \quad (5a)$$

$$\delta U(\xi,\eta) = W_s \delta T_s \quad s = 1,\ldots,5 \quad (5b)$$

where ω is the angle between the radius-vector $\mathbf{R}(\theta)$ of the contour point $\bar{C}$ and the normal to the crack contour (Fig. 1), and

$$R(\theta) = \frac{a}{\sqrt{\sin^2\theta + \lambda^2\cos^2\theta}} \quad (6a)$$

$$\cos\omega = (\sin^2\theta + \lambda^2\cos^2\theta)(\sin^2\theta + \lambda^4\cos^2\theta)^{-1/2} \quad (6b)$$

Here the quantity s is the number of the translation which retains the crack contour ellipticity; δT_s is the characteristic parameter of the s-translation; μ_s is the coefficient of the contour x-translation. According to Vainshtok and Varfolomeyev (1990), there are five linearly independent translations of the contour points. Those translations are related to the changes in the crack semi-axes ($\delta T_1 = \delta a; \delta T_2 = \delta b$); rigid displacement of the crack in the direction of the axes x and y ($\delta T_3 = \delta y; \delta T_4 = \delta x$); and finally a rotation of the crack as a unit by an angle of $\delta\varphi(\delta T_5 = \delta\varphi)$ in an anti-clockwise direction. For those translations the μ_s and W_s values have the following form:

$$\mu_1 = \sin^2 t; \quad \mu_2 = \lambda\cos^2 t; \quad \mu_3 = \sin t; \quad \mu_4 = \lambda\cos t; \quad \mu_5 = -\frac{1-\lambda^2}{2\lambda}\sin 2t \quad (7a\text{–}e)$$

$$W_1 = \frac{\partial U}{\partial a} + \frac{\lambda}{a}\frac{\partial U}{\partial \lambda}; \quad W_2 = -\frac{\lambda^2}{a}\frac{\partial U}{\partial \lambda} + \frac{\partial U}{\partial b} \quad (8a\text{–}b)$$

$$W_3 = -\frac{\partial U}{\partial \xi}; \quad W_4 = -\frac{\partial U}{\partial \eta}; \quad W_5 = -\frac{\partial U}{\partial \varphi} \quad (8c\text{–}e)$$

From the first and the second translations one can form their linear combination when both crack semi-axes vary simultaneously with $\delta a = \lambda\delta b$, i.e. the λ value does not change during translation. Designate this translation as zero one, then $\delta T_0 = \delta a$ and

$$\mu_0 = 1 \quad (7f)$$

$$W_0 = \frac{\partial U}{\partial a} + \frac{\partial U}{\lambda\partial b} \quad (8f)$$

Expressions (7) for μ_s are always valid, while expression (8) for W_s hold true for the case of uniform loading of the crack faces. The fact is that during translation the crack gets as if in another stress field with respect to its centre and axes. For the subsequent analysis we have to develop the theory of translations in a nonuniform stress field.

3. THE THEORY OF TRANSLATIONS IN A POLYNOMIAL FIELD

The aforementioned five translations can be divided into two groups: for the first two translations the crack shape and dimensions are changing; for the next three translations the crack location is changing (the positions of the centre or the axes). For the polynomial load (B) the translation of the first group results in the change of the load scale, whereas the translation of the second group changes the very law of the load action with respect to the crack centre. Consider each translation individually. Assume that the stresses expressed by law (B) are applied to the faces of the old (nontranslated) crack. The values of semi-axes, local (related to the crack centre or axes) coordinates, displacement of the surface points, etc., changed as a result of translations, will be designated in this section with subscript 1. Thus:

Translation S = 0

$$b_1 = b+\delta b \quad a_1 = a+\delta a; \quad \lambda\cdot\delta b = \delta a; \quad \lambda\cdot b = a; \quad \delta T_0 = \delta a$$

For the new crack the law of loading of its faces is given in the form:

$$p_1(x,y) = p_{ij}\left(\frac{x}{b_1}\right)^j\left(\frac{y}{a_1}\right)^i\left(\frac{b_1}{b}\right)^j\left(\frac{a_1}{a}\right)^i \approx p_{ij}\left(\frac{x}{b_1}\right)^j\left(\frac{y}{a_1}\right)^i\left(1+j\frac{\delta b}{b}+i\frac{\delta a}{a}\right), \quad (9)$$

i.e. in accordance with eqn (9) a change occurred in the law of the loading. According to Dyson's theorem new displacement $U_1(\xi_1, \eta_1)$ for loading eqn (9) are equal to:

$$U_1 = p_{ij} a_1 \Omega(a_1, b_1) \sum_{m+n=0}^{m+n=i+j} q_{m,n}^{i,j} \left(\frac{\xi}{a_1}\right)^m \left(\frac{\eta}{b_1}\right)^n \left(1 + j\frac{\delta b}{b} + i\frac{\delta a}{a}\right) \tag{10}$$

Since $\delta U = U_1 - U$, comparing eqns (C.1) and (10) we get:

$$\frac{\delta U}{\delta T} = W_0 = \frac{p_{ij}}{\Omega} \sum_{m+n=0}^{m+n=i+j} q_{m,n}^{i,j} \left(\frac{\xi}{a}\right)^m \left(\frac{\eta}{b}\right)^n + p_{ij}\Omega \sum_{m+n=0}^{m+n=i+j} q_{m,n}^{i,j} \left(\frac{\xi}{a}\right)^m \left(\frac{\eta}{b}\right)^n (i+j-m-n) \tag{11a}$$

or in another interpretation

$$\frac{\partial U}{\delta T_0} = W_0 = \frac{\partial U}{\partial a} + \frac{\partial U}{\lambda\, \partial b} + U\left(-\frac{\partial p(y,x)}{\partial a} - \frac{\partial p(y,x)}{\lambda\, \partial b}\right) \tag{11b}$$

where $U(\tilde{P}(x,y))$ is the displacement of the surface points induced by forces $\tilde{P}(x,y)$. It is evident that in the case of a uniform loading $\delta P/\delta a = \delta P/\delta b = 0$, and expression (11b) coincides with eqn (8f).

Translation S = 1

$$a_1 = a + \delta a;\quad b_1 = b;\quad \lambda_1 = \lambda + \frac{\delta a}{a}\lambda;\quad k_1 = k - \frac{\delta a}{a}\frac{\lambda^2}{k};\quad \delta T_1 = \delta a$$

The normalized law of the crack faces loading:

$$p_1(x,y) = p_{ij}\left(\frac{y}{a_1}\right)^i \left(\frac{x}{b_1}\right)^j \left(\frac{a_1}{a}\right)^i \approx p_{ij}\left(\frac{y}{a_1}\right)^i \left(\frac{x}{b_1}\right)^j \left(1 + i\frac{\delta a}{a}\right) \tag{12}$$

Considering (C.1) the displacement corresponding to loading (12) is equal to:

$$U_1(\xi_1, \eta_1) = p_{ij}\left(1 + i\frac{\delta a}{a}\right) a_1 \Omega \sum_{m+n=0}^{m+n=i+j} q_{m,n}^{i,j}(k_1) \left(\frac{\xi}{a_1}\right)^m \left(\frac{\eta}{b_1}\right)^n \tag{13}$$

Comparing new displacement (13) with the old ones (C.1), we obtain:

$$\frac{\delta U}{\delta T_1} = W_1 = \frac{p_{ij}\left(\frac{\xi}{a}\right)^2}{\Omega} \sum_{m+n=0}^{m+n=i+j} q_{m,n}^{i,j}(k) \left(\frac{\xi}{a}\right)^m \left(\frac{\eta}{b}\right)^n + p_{ij}\Omega \sum_{m+n=0}^{m+n=i+j} q_{m,n}^{i,j} \left(\frac{\xi}{a}\right)^m \left(\frac{\eta}{b}\right)^n (1 - \lambda_{m,n}^{i,j} + i - m) \tag{14a}$$

or

$$\frac{\delta U}{\delta T_1} = W_1 = \frac{\partial U}{\partial a} - \frac{\lambda^2}{ak}\frac{\partial U}{\partial k} + U\left(-\frac{\partial p}{\partial a}\right) \tag{14b}$$

where

$$\gamma^{i,j}_{m,n}(k) = \frac{\lambda^2}{kq^{i,j}_{m,n}} \cdot \frac{\partial q^{i,j}_{m,n}}{\partial k} \tag{15}$$

In the case of uniform loading, expressions (14b) and (8a) coincide.

Translation S = 2

$$a_1 = a; \quad b_1 = b - \delta b; \quad \lambda_1 = \lambda\left(1 - \frac{\delta b}{b}\right); \quad k_1 = k + \frac{\delta b}{b}\frac{\lambda^2}{k}; \quad \delta T_2 = \delta b$$

Similarly to the foregoing, we can get:

$$\frac{\delta U}{\delta T_2} = W_2 \frac{p_{ij}\left(\frac{\eta}{a}\right)^2 \lambda}{\Omega} \cdot \sum_{m+n=0}^{m+n=i+j} q^{i,j}_{m,n}(k)\left(\frac{\xi}{a}\right)^m\left(\frac{\eta}{b}\right)^n + p_{ij}\lambda\Omega$$
$$\times \sum_{m+n=0}^{m+n=i+j} q^{i,j}_{m,n}(k)\left(\frac{\xi}{a}\right)^m\left(\frac{\eta}{b}\right)^n(\gamma^{i,j}_{m,n} + j - n) \tag{16a}$$

or

$$\frac{\delta U}{\delta T_2} = W_2 = \frac{\partial U}{\partial b} + \frac{\lambda^2}{bk}\frac{\partial U}{\partial k} + U\left(-\frac{\partial p}{\partial b}\right) \tag{16b}$$

Note that in the derivation of expressions (14) and (16) it was assumed that coefficient p_{ij} are independent of λ.

Now let us consider the translations of the second group.

Translation S = 3

The crack as a unity is translated parallel to the y-axis by the δy value. Relating the new local coordinates with the new crack centre, we can write:

$$x_1 = x; \quad y_1 = y - \delta y; \quad \xi_1 = \xi - \delta y; \quad \eta_1 = \eta; \quad \delta T_3 = \delta y \tag{17}$$

By analogy, for crack translation given by (17) it can be obtained:

$$\frac{\delta U}{\delta T_3} = W_3 = p_{ij}\frac{\xi/a}{\Omega}\sum_{m+n=0}^{m+n=i+j} q^{i,j}_{m,n}\left(\frac{\xi}{a}\right)^m\left(\frac{\eta}{b}\right)^n$$
$$-p_{ij}\Omega \cdot \sum_{m+n=0}^{m+n=i+j} q^{i,j}_{m,n}\left(\frac{\xi}{a}\right)^{m-1}\left(\frac{\eta}{b}\right)^n m + p_{ij}\Omega \sum_{m+n=0}^{m+n=i+j-1} q^{i-1,j}_{m,n}\left(\frac{\xi}{a}\right)^m\left(\frac{\eta}{b}\right)^n i \tag{18a}$$

or

$$\frac{\delta U}{\delta T_3} = W_3 = -\frac{\partial U}{\partial \xi} + U\left(\frac{\partial p(x,y)}{\partial y}\right) \tag{18b}$$

Here it should be noted that the coefficients of the corresponding matrices $q^{i,j}_{m,n}$ and $q^{i-1,j}_{m,n}$ are coefficients different in both the dimensionalities and in the magnitudes.

Translation S = 4

By analogy with the third translation, we write:

$$x_1 = x - \delta x; \quad y_1 = y; \quad \xi_1 = \xi, \quad \eta_1 = \eta - \delta x; \quad \delta T_4 = \delta x \tag{19}$$

Similarly to the foregoing, we get:

$$\frac{\delta U}{\delta T_4} = W_4 = p_{ij}\frac{\lambda\eta/b}{\Omega}\sum_{m+n=0}^{m+n=i+j} q_{m,n}^{i,j}\left(\frac{\xi}{a}\right)^m\left(\frac{\eta}{b}\right)^n$$
$$-p_{ij}\lambda\Omega\sum_{m+n=0}^{m+n=i+j} q_{m,n}^{i,j}\left(\frac{\xi}{a}\right)^m\left(\frac{\eta}{b}\right)^{n-1} n + p_{ij}\lambda\Omega\sum_{m+n=0}^{m+n=i+j-1} q_{m,n}^{i,j-1} j\left(\frac{\xi}{a}\right)^m\left(\frac{\eta}{b}\right)^n \tag{20a}$$

or

$$\frac{\delta U}{\delta T_4} = W_4 = -\frac{\partial U}{\partial \eta} + U\left(\frac{\partial p(x,y)}{\partial x}\right) \tag{20b}$$

Translation S = 5

The crack as a unity is translated by an angle of $\delta\varphi$. In a new coordinate system passing through the axes of the translated crack:

$$\varphi_1 = \varphi - \delta\varphi; \quad x_1 = x + y\delta\varphi; \quad y_1 = y - x\delta\varphi;$$
$$\xi_1 = \xi - \eta\delta\varphi; \quad \eta_1 = \eta + \xi\delta\varphi; \quad \delta T_5 = \delta\varphi \tag{21}$$

For this translation similarly we get:

$$\frac{\delta U_5}{\delta T_5} = W_5 = p_{ij}a\frac{\frac{\xi}{a}\frac{\eta}{b}(\lambda^2-1)}{\lambda\Omega}\sum_{m+n=0}^{m+n=i+j} q_{m,n}^{i,j}\left(\frac{\xi}{a}\right)^m\left(\frac{\eta}{b}\right)^n$$
$$+\frac{p_{ij}a}{\lambda}\Omega\sum_{m+n=0}^{m+n=i+j} q_{m,n}^{i,j}\left(\frac{\xi}{a}\right)^{m-1}\left(\frac{\eta}{b}\right)^{n-1}\left(\left(\frac{\xi}{a}\right)^2\lambda^2 m - \left(\frac{\eta}{b}\right)^2 n\right)$$
$$+p_{ij}a\Omega\sum_{m+n=0}^{m+n=i+j} q_{m,n}^{i-l,j+1}\left(\frac{\xi}{a}\right)^m\left(\frac{\eta}{b}\right)^n\frac{i}{\lambda} - p_{ij}a\Omega\sum_{m+n=0}^{m+n=i+j} q_{m,n}^{i+1,j-1}\left(\frac{\xi}{a}\right)^m\left(\frac{\eta}{b}\right)^n j\cdot\lambda \tag{22a}$$

or

$$\frac{\delta U_5}{\delta T_5} = -\frac{\partial U}{\partial \varphi} + U\left(\frac{\partial p(\rho,\varphi)}{\partial \varphi}\right) \tag{22b}$$

or

$$\frac{\delta U_5}{\delta T_5} = -\frac{\partial U}{\partial \xi}\eta + \frac{\partial U}{\partial \eta}\xi + U\left(\frac{\partial p}{\partial y}x - \frac{\partial p}{\partial x}y\right) \tag{22c}$$

With this the development of the procedure of crack translation in a nonuniform stress field is completed.

4. THE MAIN IDEA OF THE STUDY

Consider eqn (3). Similarly to the point weight function method (see Orynyak *et al.*, 1993; Orynyak and Borodii, 1995), the loading of the crack faces by a pair unit concentrated forces applied symmetrically to the opposite crack faces in the same point with the coordinates (x, y) are taken as the first state. Then

$$p^1 = 1 \cdot \delta(x, y) \tag{23}$$

where $\delta(x, y)$ is the Dirac function a K_I^1 is the weight function or Green's function for the stress intensity factor (see Bueckner, 1970). For convenience we write:

$$K_I^1 = \frac{F(r, \varphi, t)}{\sqrt{\pi a}} \Pi^{1/4}(t) \tag{24}$$

where F is dimensionless weight function.

For the second state we consider that the displacements of the crack front points U^2 are described by eqn (C.1), then the K_I^2 values are described by eqn (D). Substituting expressions (C.1), (D), (5), (6), (23), (24) into (3), we obtain

$$\frac{1}{2}\int_\Gamma F(r, \varphi, t)\mu_S \left(\sum_{m+n=0}^{m+n=i+j} q_{m,n}^{i,j} \sin^m t \cos^n t \right) \mathrm{d}\Gamma = \frac{\delta U^2}{\delta T_S} \tag{25a}$$

or

$$\frac{1}{2}\int F(r, \varphi, t)\mu_S \bar{K}^2 \,\mathrm{d}\Gamma = \frac{\delta U^2}{\delta T_S} \tag{25b}$$

Formula (25) is the main formula of the method and the idea of the method is as follows. Two or more sets of μ_s and U_2 are prescribed so that the products $\mu_s(\theta) \cdot \bar{K}_I^2$ are equal for each of the sets. As it follows from eqn (25), in this case the right-hand sides must be equal. Taking in one of these sets the U_0^2 value for which, according to eqn (2), coefficients $p_{i,j}^{m,n}$ are known (the first of such values of U_0^2 is the known solution for U in a uniform stress field), one can determine these coefficients for another law of the crack faces displacement. With this the procedure of solution is stated.

5. EXAMPLES OF THE SOLUTION

5.1. *Crack translation in a uniform stress field*

Using translations 1–5 in succession, considering eqn (4) we get from eqn (25):

$$\frac{1}{2}\int_\Gamma F \sin^2\theta \,\mathrm{d}\Gamma = (1-\gamma_{0,0}^{0,0})\Omega + \frac{\xi^2}{a^2\Omega}; \quad \frac{1}{2}\int_\Gamma F\cos^2\theta \,\mathrm{d}\Gamma = \gamma_{0,0}^{0,0}\Omega + \frac{\eta^2}{b^2\Omega} \tag{26a–b}$$

$$\frac{1}{2}\int_\Gamma F\sin\theta \,\mathrm{d}\Gamma = \frac{\xi}{a\Omega}; \quad \frac{1}{2}\int_\Gamma F\cos\theta \,\mathrm{d}\Gamma = \frac{\eta}{b\Omega}; \quad \frac{1}{2}\int_\Gamma F\cos\theta\sin\theta \,\mathrm{d}\Gamma = \frac{\eta\xi}{ba\Omega} \tag{26c–e}$$

5.2. *Linear law of the crack faces displacement*

We prescribe the following law of the crack faces displacement:

$$U = a\frac{\xi}{a}\Omega \tag{27}$$

In accordance with Dyson's theorem [eqns (1) and (2)], the following law of the crack faces loading fits it:

$$p(x,y) = p_{0,0}^{1,0} + p_{1,0}^{1,0}\left(\frac{y}{a}\right) + p_{0,1}^{1,0}\left(\frac{x}{b}\right) \tag{28a}$$

Since $p_{0,0}^{1,0}q_{0,0}^{0,0}+p_{1,0}^{1,0}q_{0,0}^{1,0}+p_{0,1}^{1,0}q_{0,0}^{0,1} = 0$ [otherwise the displacements would not have taken the form (27)], applying a zero translation to eqn (27) according to formula (11a), using formula (25) and taking into account the dimensionless $\bar{K}_1$ value corresponding to eqn (27), we get:

$$\frac{1}{2}\int F\sin\theta\,\mathrm{d}\Gamma = \frac{\xi}{a\Omega} - p_{0,0}^{1,0}q_{0,0}^{0,0}\Omega \tag{29}$$

Comparing eqns (29) and (26c), it is evident that $p_{0,0}^{1,0} = 0$. Apply the fourth translation to eqn (27) according to eqn (20b). Using expressions (25) and (26c) we similarly get that $p_{0,1}^{1,0} = 0$.

Thus we have demonstrated that if the displacements are prescribed in the form of eqn (27), the following loading corresponds to them

$$p(x,y) = p_{1,0}^{1,0}\left(\frac{y}{a}\right) \tag{28b}$$

Similarly, it can be shown that if

$$U(\xi,\eta) = a\frac{\eta}{b}\Omega \tag{30a}$$

then

$$p(x,y) = p_{0,1}^{0,1}\frac{x}{b} \tag{30b}$$

To definite coefficient $p_{0,1}^{1,0}$, we shall apply the third translation to expression (27). In accordance with eqn (18b), we have:

$$\frac{\delta U}{\delta T_3} = -\Omega + \frac{\xi^2/a^2}{\Omega} + \frac{\Omega}{E(k)}p_{1,0}^{1,0} \tag{31}$$

Substituting (31) into eqn (25) for $S = 3$ we get that the left-hand sides of the expression obtained and of (26a) are equal. Therefore, comparing their right-hand sides and taking into account eqn (15) and the fact that $q_{0,0}^{0,0} = 1/E(k)$, we get:

$$q_{1,0}^{1,0} = \frac{1}{p_{1,0}^{1,0}} = \frac{k^2}{E(k)(1+k^2)-(1-k^2)K(k)} \tag{32}$$

where $K(k)$ is the complete elliptic integral of the first kind. The result obtained agrees with the known solution of Shah and Kobayashi (1971).

To determine coefficient $p_{0,1}^{0,1}$, apply to expression (30a) the fourth translation according to eqn (20b). Substituting all necessary parameters into eqn (25b) and comparing the result with eqn (26b), we get

$$q_{0,1}^{0,1} = \frac{1}{p_{0,1}^{0,1}} = \frac{k^2}{(1-k^2)K(k)+(2k^2-1)E(k)} \tag{33}$$

Expression (33) coincides with the known solution of Shah and Kobayashi (1971).

For later use we shall need expressions like (26) and those set up from the known solutions for the crack faces displacements described by formulas (27) and (30a). As it was mentioned above, the methodology of the crack translation devised in Section 3 (for $S = 1$ and $S = 2$) is valid when the coefficient $p_{i,j}^{m,n}$ are independent of λ. Hence, rewrite the obtained equations once again:

$$p(x,y) = \frac{y}{a} \Rightarrow U = q_{1,0}^{1,0} a \frac{\xi}{a} \Omega \tag{34a}$$

$$p(x,y) = \frac{x}{b} \Rightarrow U = q_{0,1}^{0,1} a \frac{\eta}{b} \Omega \tag{34b}$$

Here coefficient $q_{1,0}^{1,0}$ and $q_{0,1}^{0,1}$ have been defined by formulas (32) and (33). For the crack eqn (34a) we make the first translation in accordance with eqn (14b) and substitute the result into formula (25). Considering eqns (15) and (34a) we get:

$$\frac{1}{2}\int_\Gamma F \sin^3\theta \, d\Gamma = \frac{\xi}{a}\Omega + \frac{\xi^3}{a^3\Omega} - \frac{\xi}{a}\Omega\gamma_{1,0}^{1,0} \tag{35a}$$

Subtracting eqn (35a) from expression (26c), we get:

$$\frac{1}{2}\int_\Gamma F \sin\theta\cos^2\theta \, d\Gamma = \frac{\xi}{a}\frac{\eta^2}{b^2\Omega} + \frac{\xi}{a}\gamma_{1,0}^{1,0}\Omega \tag{35b}$$

Similarly, making the first translation for crack eqn (34b), we get:

$$\frac{1}{2}\int_\Gamma F \cos\theta\sin^2\theta \, d\Gamma = \frac{\eta}{b}\Omega + \frac{\eta}{b}\frac{\xi^2}{a^2\Omega} - \frac{\eta}{b}\Omega\gamma_{0,1}^{0,1} \tag{35c}$$

Subtracting expression (35c) from eqn (26d), we obtain:

$$\frac{1}{2}\int_\Gamma F \cos^3\theta \, d\Gamma = \frac{\eta^3}{b^3\Omega} + \frac{\eta}{b}\Omega\gamma_{0,1}^{0,1} \tag{35d}$$

5.3. *The law of the crack faces displacement is proportional to the product* ($\xi \times \eta$)

Let:

$$U(\xi,\eta) = a\frac{\xi}{a}\frac{\eta}{b}\Omega \tag{36a}$$

In the general case, the following law of the crack faces loading corresponds to those displacements:

$$p(x,y) = p_{0,0}^{1,1} + p_{1,0}^{1,1}\left(\frac{y}{a}\right) + p_{0,1}^{1,1}\left(\frac{x}{b}\right) + p_{1,1}^{1,1}\left(\frac{x}{b}\frac{y}{a}\right) + p_{2,0}^{1,1}\left(\frac{y}{a}\right)^2 + p_{0,2}^{1,1}\left(\frac{x}{b}\right)^2 \tag{36b}$$

Applying a zero translation to eqn (36a) according to formula (11b), substituting the results into eqn (25) and comparing with eqn (26e), we obtain:

$$p^{1,1}_{0,0} = p^{1,1}_{1,0} = p^{1,1}_{0,1} = 0 \tag{37a}$$

Now applying the fourth translation to eqn (36a) according to formula (20b), substituting the result into eqn (25) and comparing with eqn (35b), we get:

$$p^{1,1}_{0,2} = 0 \tag{37b}$$

$$p^{1,1}_{1,1} = E\frac{2k^4-2k^2+2}{k^4} - K\frac{(1-k^2)(2-k^2)}{k^4} \tag{37c}$$

Note that by using the third translation for eqn (36a), similarly we can get:

$$p^{1,1}_{2,0} = 0 \tag{37d}$$

Therefore, the only nonzero coefficient in expansion (36b) is determined by eqn (37c) which fully agrees with the known solutions of Shah and Kobayashi (1971) and Shail (1978).

5.4. *One formal generalization*

In previous examples we have obtained many zero coefficients. Let us generalize this result in order to simplify the form of representing the polynomials sought. Thus:

Only if the pairs of numbers i and m; j and n are the numbers of similar evennes ("0" is considered to be even), the coefficients $q^{i,j}_{m,n}$ and $p^{m,n}_{i,j}$ are not equal to zero. In other words:

$$\text{if} \quad \{|i-m| = 2z_1+1 \quad or \quad |j-n| = 2z_2+1\}, \quad \text{then} \quad q^{i,j}_{m,n} = p^{m,n}_{i,j} \equiv 0 \tag{38}$$

where z_1 and z_2 are whole numbers 0, 1, 2, ...

Condition eqn (38) corresponds to the analysis of Shah and Kobayashi (1971), Borodachev (1981), etc.

5.5. *Quadratic law of the crack faces displacement*

Let:

$$U(\xi,\eta) = a\frac{\xi^2}{a^2}\Omega; \quad U(\xi,\eta) = a\frac{\eta^2}{b^2}\Omega \tag{39a–b}$$

In order to find the corresponding loading coefficients we apply the zero, the fourth and third translations to eqn (39). Comparing the results obtained for eqn (39a) with expressions (26a), (35c), (35a), and similarly the results obtained for eqn (39b) with expressions (26b), (35d), (35b), we get:

$$p^{2,0}_{0,0} = -\frac{1-\gamma^{0,0}_{0,0}}{2\cdot q^{0,0}_{0,0}}; \quad p^{2,0}_{2,0} = \frac{3-\gamma^{1,0}_{1,0}}{2\cdot q^{1,0}_{1,0}}; \quad p^{2,0}_{0,2} = \frac{1-\gamma^{0,1}_{0,1}}{2\cdot q^{0,1}_{0,1}} \tag{40a–c}$$

and

$$p^{0,2}_{0,0} = -\frac{\gamma^{0,0}_{0,0}}{2q^{0,0}_{0,0}}; \quad p^{0,2}_{0,2} = \frac{2+\gamma^{0,1}_{0,1}}{2q^{0,1}_{0,1}}; \quad p^{0,2}_{2,0} = \frac{\gamma^{1,0}_{1,0}}{2q^{1,0}_{1,0}} \tag{41a–c}$$

Note that the coefficients of the matrix

$$[Q] = \begin{pmatrix} q^{0,0}_{0,0} & 0 & 0 \\ q^{2,0}_{0,0} & q^{2,0}_{2,0} & q^{2,0}_{0,2} \\ q^{0,2}_{0,0} & q^{0,2}_{2,0} & q^{0,2}_{0,2} \end{pmatrix} \tag{42}$$

are related to the coefficients of the matrix

$$[P] = \begin{pmatrix} p^{0,0}_{0,0} & 0 & 0 \\ p^{2,0}_{0,0} & p^{2,0}_{2,0} & p^{2,0}_{0,2} \\ p^{0,2}_{0,0} & p^{0,2}_{2,0} & p^{0,2}_{0,2} \end{pmatrix} \tag{43}$$

in the following way:

$$[Q] = [P]^{-1} \tag{44}$$

This allows $q^{i,j}_{m,n}$ to be calculated from the known $p^{m,n}_{i,j}$. For quadratic law of loading considered they are equal to:

$$q^{2,0}_{0,0} = \frac{E^2(7k^2-5)+EK(1-k^2)(8-6k^2)-K^2 3(1-k^2)^2}{6\Delta(k)E} \tag{45a}$$

$$q^{0,2}_{0,0} = \frac{E^2(1-k^2)(-2k^2-5)+EK(1-k^2)(8-2k^2)-K^2 3(1-k^2)^2}{6\Delta(k)E} \tag{45b}$$

$$q^{2,0}_{2,0} = \frac{E^2(6k^4-2k^2-2)+EK(1-k^2)(2+3k^2)}{3\Delta(k)E} \tag{45c}$$

$$q^{0,2}_{2,0} = \frac{E^2(k^2-1)(2-k^2)+EK2(1-k^2)^2}{3\Delta(k)E} \tag{45d}$$

$$q^{2,0}_{0,2} = \frac{E^2(k^2-2)+EK2(1-k^2)}{3\Delta(k)E} \tag{45e}$$

$$q^{2,0}_{0,2} = \frac{E^2(2k^4+6k^2-2)-EK(1-k^2)(2+k^2)}{3\Delta(k)E} \tag{45f}$$

where

$$\Delta(k) = \frac{E^2(-11+11k^2+4k^4)+EK(1-k^2)(16-8k^2)-K^2 5(1-k^2)^2}{2} \tag{46}$$

6. CONCLUDING REMARKS

In the present paper we have not made it our goal to obtain new values for K_{I}. The main result is that a new method is proposed which is appreciably simpler than the known methods. It does not require the boundary problem solution and actually represents itself the recurrent procedure. Suffice to say that the only technical complexities of the method are: differentiation of complete elliptic integrals:

$$\frac{\partial E}{\partial k} = \frac{E}{k} - \frac{K}{k} \quad \frac{\partial K}{\partial k} = \frac{E}{(1-k^2)k} - \frac{K}{k} \tag{47}$$

and transformation of the matrices by formula (44). This makes the method accessible virtually for every engineer. A general scheme of the application of the method is as follows:

(1) For the known solutions the first and the second translations of the crack are made. By substituting the required expressions into formula (25) the reference integrals are determined.
(2) The unknown coefficients $p_{i,j}^{m,n}$ in the expansion of the loading for the chosen form of displacements are determined by carrying out the zero, the third and fourth translations of the crack, substituting the corresponding values into integral formula (25) and comparing the obtained integrals with the reference ones.
(3) Inverse coefficients $q_{m,n}^{i,j}$ are defined from formula (44) and again a transfer is made to procedure 1.

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